MCQs of Plasma Physics

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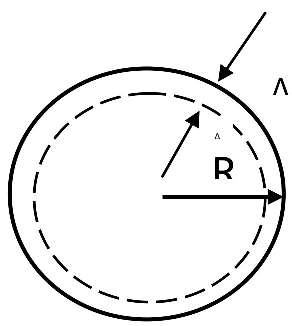
Lecture 1

Problem 1: Consider a singly ionized sphere of electron density n_o, radius R and electron temperature T. Due to thermal motions electrons of the layer of width Δ near the surface move out (leaving ions behind) such that the potential energy of an electron at the surface equals in magnitude the average thermal energy 3T/2 (where Botzmann's Constant in hidden in T). Estimato Δ.

Solution: The number of electrons in the surface layer of width Δ and surface area $4\pi R^2$ is

$$N=4\pi R^2 n_o \Delta.$$

Once they move out the sphere, upto radius R, has net positive charge



Q = Ne , where –e is the electron charge. The Potential at r =R is

$$\varphi\,(R) = \frac{Q}{4\pi\varepsilon_0 R}.$$

The magnitude of potential energy of an electron at the surface is

$$e\varphi = \frac{(4\pi R^2 \Delta n_o \Delta e^2)}{4\pi\epsilon_o R} = \frac{3}{2}\,T$$

$$\Delta = \frac{3}{2}\,\frac{\epsilon_0 T}{n_o\, e^2 R} = 3\lambda_D^2\,/\,R,$$

where λ_D is the Debye length. For a sphere of Radius R>λ_D, when the electrons move out a distance shorter than Debye length, the space change field stops the electrons to move further. This space charge field pulls the ions radially outward causing ambipolar expansion of the sphere where electrons and ions move almost together.

Lecture 2

Problem 1: Consider a plasma with a relativistic electron beam of rest mass m, charge –e, drift velocity $\vec{v}$ and momentum $\vec{p}$. Deduce the fluid equation of motion for beam electrons in the presence of electric field $\vec{E}$ and magnetic field$\vec{B}$. Neglect collisions and thermal motions.

Solution: The equation of motion is

$$\frac{\overrightarrow{dp}}{dt} = -e\left(\vec{E} + \vec{v}\ x\ \vec{B}\right)$$

$$\frac{\overrightarrow{dp}}{dt} = Lt_{\Delta t \to 0}\ \frac{\vec{p}(\vec{r}+\overrightarrow{\Delta r}, t+\Delta t) - \vec{p}\ (\vec{r}, t)}{\Delta t}$$

$$= \frac{\partial \vec{P}}{\partial t} + \vec{v}.\nabla \vec{p}$$

Further $\vec{p} = m\gamma\vec{v}$ where $\gamma = (1+p^2/m^2c^2)^{1/2} = (1-v^2/c^2)^{-1/2}$ Hence the equation of motion is

$$m\left[\frac{\partial(\gamma\vec{v})}{\partial t} + \vec{v}.\nabla(\gamma\,\vec{v})\right] = -\,e\vec{E} - e\vec{v}\ \vec{E}\ x\vec{B}\ .$$

Problem 2: A singly ionized helium plasma at temperature T_0 has electron collision. frequency ν=10^4 sec^{-1} . When temperature is raised to 9 T_0 it becomes fully ionized, Estimate the collision frequency ignoring the change in the logarithmic factor.

Solution : $\nu \approx c_1 n_i z_i^2 T_e^{-3/2}$

where c_1 is constant and z_i is the ion charge number. For z_i =1, T_e =T_o,

$$\nu \approx c_1 n_i\ T_o^{-3/2} = 10^4\ s^{-1}$$

When plasma is fully ionized z_i = 2, T_e=9 T_0,

$$\nu \approx c_1 n_i \, 4 \, T_0^{-3/2} / 27$$

$$= \frac{4}{27} \text{ x } 10^4 \text{ s}^{-1} .$$

Problem 3: Estimate the mean free path of electrons in a weakly ionized plasma of electron –atom collision cross – section q= 10^{-15} cm^2, temperature T_e = 1eV and plasma pressure 10^{-3} Torr. Take neutral particle temperature T_n=0.1 eV, 1 Torr =0.13 N/m^2.

Solution :

Pressure of neural atoms of density n_n is

$$P = n_n T_n = 0.13,$$

$$n_n = \frac{0.13}{0.1 \text{ x } 1.6 \text{ x} 10^{-19}} = 8 \text{ x } 10^{18} \text{ m}^{-3} \,,$$

$$\text{Mean free path } \lambda_m = 1/n_n q = 1.2 \text{ m.}$$

Lecture 3

Problem 1: A dc electric field field $E\hat{x}$ is applied at t=0 to a plasma of electron density n_o and collision frequency ν. How would the current density evolve with time?

Solution:

The equation of motion for electrons is

$$\frac{\partial \vec{v}}{\partial t} + \nu \vec{v} = - \frac{e \vec{E}}{m}$$

Multiplying by $-n_o e$ and writing $\vec{J} = -n_o e \vec{v}$, we obtain

$$\frac{\partial \vec{J}}{\partial t} + \nu \vec{J} = \frac{n_o e^2}{m} \vec{E}$$

Multiply both sides by exp (νt) , rewrite this equation as

$$\frac{\partial}{\partial t}\left(\vec{J}\ e^{\nu t}\right) = \frac{n_o e^2}{m} \vec{E}\ e^{\nu t}$$

and integrate,

$$\vec{J} = \frac{n_o e^2}{m} e^{\nu t} \int_{-\infty}^{t} \vec{E}\ e^{\nu t}\, dt$$

$$= \frac{n_o e^2}{m} \vec{E}\ (1 - e^{\nu t}).$$

$\vec{J}$ is zero at t=0 and rises to the steady state value $\frac{n_o e^2 \vec{E}}{m\nu}$ on the time scale of ν^{-1} ,the collision time.

Problem 2 : How does the dc conductivity of a strongly ionized plasma increase with electron density when electron temperature is kept constant? Ignore the variation in logarithmic factor in collision frequency.

Solution:

$$\sigma = \frac{n_o e^2}{mv} \ .$$

As v is proportional to n_o, (the electron density), σ remains constant as n_o increases.

Problem 3: A fully ionized plasma sphere of initial radius R_0, electron density n_o and temperature T_0 undergoes ambipolar expansion. How would the electrical conductivity change with radius R?

Solution: In adiabatic expansion temperature and volume (for monatomic gas of specific heats ratio γ=5/3) go as

$$T\ v^{r-1} = \text{const.}$$

$$T \sim v^{-(r-1)} \sim R^{-3(r\ -1)} = R^{-2}\ ,$$

$$\sigma \sim T^{3/2} \sim R^{-3}.$$

The conductivity increases as R^{-3}.

Lecture 4

Problem 1: Estimate radio frequency ω at which the displacement current and conduction current in a collisionless plasma are equal in magnitude

Solution:

Conduction current density $\vec{J} = \sigma \ \vec{E} = \frac{-n_o}{m\, i\omega} e^2 \ \vec{E}$

Displacement current density $\overrightarrow{J_D} = \frac{\partial \vec{D}}{\partial t} = -i\omega\varepsilon_o \vec{E}$

$|\vec{J}| = |\overrightarrow{J_0}|$ gives

$$\omega = \left(\frac{n_o e^2}{m\varepsilon_0}\right)^{t/2} = \omega_p$$

At $\omega=\omega_p$ the displacement current due to an electromagnetic wave is equal and opposite to the conduction current.

Problem 2: A collisionless plasma is subjected to amplitude modulated radio field $\vec{E} = \overrightarrow{A_o} \ (1 + \ \mu \ \cos(\Omega t)_{e^{-i\omega t}}$; where $\Omega \ll \omega$. Obtain the current density $\vec{J}(t)$.

Solution :

$$\sigma = i\frac{e^2 n_o}{m\omega} \, , \ \vec{A} = \overrightarrow{A_o}\,(1 + \mu \cos \Omega t)$$

$$\overrightarrow{J(t)} = i\,\frac{n_o e^2}{m\omega}\,\vec{E} + \ \frac{n_o e^2}{m\omega^2}\,\frac{\partial \vec{A}}{\partial t}\,e^{i\omega t}$$

$$= \left[i\ \frac{n_o e^2}{m\omega}\ (1 + \mu \cos \Omega t) - \frac{n_o e^2}{m\omega^2}\ \Omega\ \mu\ \sin(\Omega t)\right].\ \overrightarrow{A_o}\ e^{i\omega t}$$

Lecture 5

Problem 1: Estimate the cyclotron frequency and Larmor radius of a 2 MeV electron in a magnetic field $\overrightarrow{B_s} = \hat{z}$ 0.5 Tesla when the pitch angle $\theta = \tan^{-1}(v_1\ /v_z)$=1

Solution : In a static magnetic field, electron kinetic energy ε is a constant of motion, only mass becomes rest mass m times the relativistic γ factor,

$$\gamma = 1 + \frac{\epsilon}{mc^2} \approx 1 + 2\ \frac{\text{Mev}}{0.5\ \text{Mev}} = 5$$

Electron speed v is

$$v\ = c\left(1 - \frac{1}{r^2}\right)^{\mu 2}\ ;$$

$$v^2 = v_\perp^2 + v_z^2 = 2v_\perp^2;$$

$$v_1 = \frac{c}{\sqrt{2}}\left(1 - \frac{1}{25}\right)^{\mu_2} = \frac{\sqrt{12}}{5}\ c$$

Cyclotron frequency ω_c and Larmor radius ρ are

$$\omega_c = \frac{e\ B_s}{m\gamma} = 1.76 \times 10^{10}\ \text{rad/s}$$

$$\rho = \frac{v_1}{\omega_c} = 1.2\ \text{cm}\ .$$

Problem 2 : An electron is born, via ionization of an atom, at the origin at t=0 with velocity $\vec{v}$= vcos $\theta\ \hat{z}$+vsin $\theta\ \ \hat{x}$, Obtain its guiding center coordinates x_g , y_g .

Solution :

$$\mathbf{x} = x_g + \frac{v_\perp}{\omega_c} \sin\left(\omega_c t + \delta\right)$$

$$y = y_g - \frac{v_{\perp 1}}{\omega_c} \cos\left(\omega_c t + \delta\right)$$

$$v_x = v_\perp \cos\left(\omega_c t + \delta\right)$$

$$v_y = v_\perp \sin\left(\omega_c t + \delta\right)$$

Using in these equations, at t=0, x=0, y=0, $v_x = v\sin\theta$, v_y = 0, one obtains

$$v_\perp = v\sin\theta \text{ ,}\delta\text{=0, } x_g\text{=0, } y_g = \frac{v_\perp}{\omega_c} \text{ .}$$

Problem 3 : A collisionless singly ionized helium plasma with static magnetic field $B_s\,\hat{z}$ is subjected to an RF field $\vec{E} = \hat{x}$ A exp (-iωt) of ω = 2 ω_{ci} , where ω_{ci} is the ion cyclotron frequency. Estimate the ratio of Hall current density J_y to Pederson current density J_x.

Solution :

$$J_x = \sigma_{xx}\,\boldsymbol{E_x} \text{ , } J_y = -\sigma_{xy}\,\boldsymbol{E_x}$$

At ω=2$\boldsymbol{\omega_{ci}}$ << $\boldsymbol{\omega_c}$, σ_{xx} and σ_{xy} in a plasma of electron density n_0, electron mass m and ion mass m_i , would be the sum of electron terms and ion terms. The latter can be obtained from electron conductivity components by replacing m by m_i, ω_c by -ω_{ci}'

$$\sigma_{xx} = -i\,\frac{n_o e^2\omega}{m\omega_c^2} + i\,\frac{n_o e^2\omega}{m_i(\omega^2-\omega_{ci}^2)}$$

$$\approx i\,\frac{2n_o e^2}{3m_i\omega_{ci}}$$

$$\sigma_{xy} = -\frac{n_o e^2}{m\omega_c} - \frac{n_o e^2\omega_{ci}}{m_i(\omega^2-\omega_{ci}^2)}$$

$$= -\frac{n_o e^2}{m_i\omega_{ci}} - \frac{n_o e^2}{3m_i\omega_{ci}} = -\frac{4n_o e^2}{3m_i\omega_c}$$

$$J_y/J_x = -\sigma_{xy}/\sigma_{xx} = -\,2i.$$

Lecture 6

Problem 1: A plasma with electron collision frequency v and dc magnetic field $B_s\hat{z}$ is subjected to RF field $\vec{E} = A(\hat{x} + i\,\hat{y} + \propto\,\hat{z})\,e^{-i\omega t}$, with ω = 1.1 ω_c , where ω_c is the electron cyclotron frequency. Obtain the electron heating rate. Take $v^2 << \omega^2$, $(\omega - \omega_c)^2$.

Solution :

Solving the equation of motion, electron drift velocity turns out to be

$$\vec{v} = \vec{v}_\perp + v_z\,\hat{z}\ ,$$

$$\vec{v}_\perp = \frac{eA\,(\hat{x} + i\hat{y})}{mi\,(\omega - \omega_{c+iv})}\,e^{-i\omega t}$$

$$v_z = \frac{e\propto A}{mi(\omega + iv)}\,e^{-i\omega t}$$

The heating rate for electrons is

$$H_e = \frac{-e}{2}\,\text{Re}\,\left[\overrightarrow{E^*}\,.\,\vec{v}\right]$$

$$= \frac{1}{2}\left[\frac{e^2A^2\alpha^2 v}{m\omega^2} + \frac{2e^2A^2 v}{m(\omega - \omega_c)^2}\right]$$

$$= \frac{e^2A^2 v}{m\omega_c^2}\left(\frac{\alpha^2}{2.42} + 200\right).$$

Problem 2: A singly ionized helium plasma with static magnetic $B_s\hat{z}$, electron collision frequency v and ion collision frequency v_i , is subjected to RF field $\vec{E} = \hat{x}$ A exp (-iωt) at ω=1.1ω_{ci}, where ω_{ci} is the ion cyclotron frequency, $(\omega - \omega_{ci},)^2 >> v_i^2$ and $\omega_c >> v$. obtain the heating rates of electrons and ions..

Solution : The heating rates of electrons and ions are

$$H_e = -\frac{e}{2}\,\text{Re}\,[E_x^* v_x] = \frac{e^2 v A^2}{2m\omega_c^2}$$

$$H_i = \frac{e}{2}\,\text{Re}\,[E_x^* v_{ix}]$$

$$=\frac{e^2A^2}{2m_i}\,\mathrm{Re}\left[\frac{v_i-i\omega}{\omega_{ci}^2-(\omega+iv_i)^2}\right]$$

$$=\frac{e^2A^2v_i\left(\omega_{ci}^2+\omega^2\right)}{\boldsymbol{2m_i}\left(\omega_{ci}^2-\omega^2\right)^{\mathbf{2}}}\ \mathbf{=25}\ \frac{e^2A^2v_i}{\boldsymbol{m_i}\,\omega_{ci}^2}\ \mathbf{.}$$

The ion heating rate is much larger than the electron heating rate due to ion cyclotron resonance effect.

Problem 3 : In problem 2 if the field were LCP (left circularly polarized), $\vec{E}\ (\hat{x}-i\hat{y})A$ exp(-iωt), estimate the electron and ion heating rates.

Solution : For LCP field, the drift velocities of electrons and ions are:

$$\vec{v}=\frac{e\,\vec{E}}{mi\ (\omega+\omega_c+iv)}$$

$$\vec{v_i}=\frac{e\,\vec{E}}{m_i\ i(\omega-\omega_{ci}+iv_i)}$$

The electron and ion heating rates are:

$$H_e = -\frac{e}{2}\,\mathrm{Re}\left[\overrightarrow{E^*}.\overrightarrow{v_i}\right]$$

$$=\frac{e^2vA^2}{m\omega_c^2}$$

$$H_i = \frac{e}{2}\,\mathrm{Re}\left[\overrightarrow{E^*}.\overrightarrow{v_i}\right] = \frac{e^2v_iA^2}{\boldsymbol{m_i}\left(\omega-\omega_{ci}\right)^{\mathbf{2}}}$$

$$=100\ \frac{e^2v_iA^2}{\boldsymbol{m_i}\,\omega_{ci}^2}$$

Lecture 8

Problem 1 : A collisionless plasma has a non - relativistic electron beam of density n_0 propagating with velocity $v_o\hat{z}$. A low amplitude electromagnetic wave propagates through it with electric field $\vec{E} = \hat{x} A\, e^{i(\omega t - kz)}$.Obtain the perturbed beam velocity.

Solution :

The equation of motion is

$$m\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}.\nabla\,\vec{v}\right) = -e\vec{E} - e\vec{v} \times \vec{B}.$$

$\vec{B}$,the magnetic field of the wave, can be written as

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = \hat{y}\ \frac{Ak}{\omega}\ e^{-i(\omega t - kz)}.$$

Expressing $\vec{v} = v_o\hat{z} + \overrightarrow{v_\perp}$ with $v_i << v_o$, the linearized equation of motion is

$$\frac{\partial \overrightarrow{v_1}}{\partial t} + v_o \frac{\partial \overrightarrow{v_1}}{\partial z} = -\frac{e\,\vec{E}}{m} = \frac{e\vec{E}}{m} - \frac{e\overrightarrow{v_0}}{m} \times \left(\vec{k} \times \vec{E}\right)$$

$$-i(\omega - kv_o)\overrightarrow{v_1} = -\frac{e\vec{E}}{m}\left(1 - \frac{kv_o}{\omega}\right)$$

$$\overrightarrow{v_1} = \frac{e\vec{E}}{mi\omega}.$$

If the equilibrium beam velocity v_o were relativistic to them would be replaced by $m\gamma = m\,(1 - v_o{}^2/c^2)^{-1/2}$.

Problem 2: A plasma cluster of radius r_c and free electron density n_o is subjected to laser field $\overrightarrow{E_L} = \hat{x}\, A_L \exp\,[-i(\omega t - k_L z)]$ where $r_c k_L << 1$. Obtain the drift velocity of cluster electrons.

Solution :

As the electron cloud of the cluster moves a distance $\vec{\Delta}$ under the electric force -e $\overrightarrow{E_L}$, with respect to ion sphere , the electrons experience space charge restoration force = - $\frac{m\omega_p^2}{3}\vec{\Delta}$ where $\omega_p = (n_o e^2/m\varepsilon_o)^{1/2}$: Thus the equation of motion for cluster electrons is

$$\frac{d^2\Delta}{dt^2} + \left(\omega_p^2 \quad /3\right)\Delta = \frac{eA_L}{m} \mathrm{L}\ e^{-i(\omega t - k_L z)}\ .$$

For v_z << c, Δ << 2Л/k one may approximate dΔ/dt = $\partial\Delta/\partial t$ = - iωΔ. Thus

$$\vec{\Delta} = + \frac{e\overrightarrow{E_L}}{m(\omega^2 - \omega_p^2/3)}$$

$$\vec{\mathrm{v}} = \frac{d\vec{\Delta}}{dt} = - \frac{i\ e\ \overrightarrow{E_L}\omega}{m(\omega^2\omega_p^2/3)}\ .$$

Problem 3: Estimate the skin depth of 10 GHz microwave in a conductor of relative lattice permittivity $\in_r$ = 4, plasma frequency $\omega_p = 10^{16}$ rad/s and collision frequency ν=$10^{12}\ s^{-1}$.

Solution :

$$\mathrm{k} = k_r + ik_i = \frac{\omega}{c}\left(\varepsilon_r - \frac{\omega_\rho^2}{\omega(\omega + i\nu)}\right)^{1/2}$$

$$\approx \frac{\omega_p}{c}\left(\frac{\omega}{2\nu}\right)^{1/2}(1+\mathrm{i})$$

$$\text{Skin depth } \delta = {k_i}^{-1} = \frac{c}{\omega_p}\left(\frac{2\nu}{\omega}\right)^{1/2}$$

$$\approx 0.17\ \mu m$$

Note that the skin depth decreases with the frequency of the wave as $\omega^{-1/2}$.

Problem 4 : Estimate the intensity of 1 μm laser that would raise the electron temperature from T_o = 1 KeV to T_e=2 KeV in a plasma. Take energy loss per collision = 3 (m/m_i) (T_e-T_o), m_i/m=3600 , ω_p/ω=0.1, collision frequency much less than the laser frequency ($\nu \ll \omega$).

Solution :

$$\text{Laser field } \vec{E} = \hat{x}\, A_o\, e^{-i(\omega t - kz)}$$

$$\text{Electron velocity } \vec{v} = \frac{e\,\vec{E}}{mi(\omega + i\nu)}$$

$$\text{Heating rate } H_e = -\frac{e}{2}\,\text{Re}\left[\vec{E^{x}}.\vec{v}\right]$$

$$= \frac{e^2|E|^2\nu}{2m\omega^2}$$

Equating H_e to energy loss rate,

$$\frac{e^2|E|^2\nu}{2m\omega^2} = 3\frac{m\nu}{m_i}\,(T_e - T_o),$$

$$T_e = T_o + \frac{e^2|E|^2 m_i}{6\,m^2\,\omega^2}\,.$$

$$\text{Laser intensity } S_{av} = \frac{|E|^2}{2\mu_o c}\,(1 - \omega_p^2/\omega^2)^{1/2}$$

$$\approx |E|^2 / 2\mu_0 c.$$

Requisite intensity to raise T_e from 1 KeV to 2 KeV is

$$S_{av} = \frac{6m^2\omega^2\left(1 - \omega_p^2/\omega^2\right)^2}{2m_i e^2\mu_o c}\;T_o \approx 5.5 \times 10^{16}\ \text{W/m}^2$$

Problem 5 : A laser , normally incident on a plasma , suffers 100 % power reflection with a phase change of π /2 . Estimate the plasma frequency in terms of laser frequency $\omega_{L.}$

Solution :

100% power reflection occurs only when $\omega_p > \omega_L$ and collision frequency ν =0. Writing η=iα , α = $\left(\frac{\omega_p^2}{\omega_L^2} - 1\right)^{1/2}$, one may write the amplitude reflection coefficient

$$R_A = \frac{1-\eta}{1+\eta} = e^{-2\,i\,\tan^{-1}\alpha}$$

$$2\tan^{-1}\alpha = \pi/2 \text{ , } \alpha=1$$

$$\omega_p = \sqrt{2}\ \omega_L.$$

Problem 6: An electromagnetic wave of frequency ω and intensity 1W/m^2 is normally incident on a collisionless plasma of plasma frequency ω_p =2ω. Estimate the amplitude of the wave at a depth of c/ω in the plasma.

Solution :

$$k = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = i\sqrt{3}\ \omega/c \text{ .}$$

Amplitude transmission coefficient

$$T_A = \frac{2}{1+kc/\omega} = \frac{2}{1+i\sqrt{3}} \text{ ‘}$$

$$|T_A| = 1 \text{ .}$$

Amplitude of the incident wave A_o is related to intensity S_{av} as

$$S_{av} = \frac{A_0^2}{2\,\mu_o c} \text{ ,}$$

$$A_0 = 2\mu_o c\ {s_{av}}^{1/2} = 27.2 \text{ V/m .}$$

Transmitted wave amplitude

$$A_T = |T_A| A_0 \; e^{-\sqrt{3}\,\omega z/c}$$

$$= 27.2 \; e^{-\sqrt{3}} \;\; \approx 5.5 \text{ V/m.}$$

Problem 7: A microwave pulse of carrier frequency 20GHZ takes 5ns to travel a distance of 1 meter in a weakly collisional plasma and suffers 3db power loss. Estimate the electron density and collision frequency of the plasma.

Solution: In a weakly collisional plasma

$$\omega^2 = \omega_p^2 + k^2 c^2,$$

$$\text{group Velocity } V_g = \frac{\partial \omega}{\partial k} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\mu 2} = \frac{1\text{m}}{5\text{ns}} = 2 \times 10^8 \text{ m/s}$$

$$\omega p = 50\sqrt{n_e} = \frac{\sqrt{5}}{3}\,\omega.$$

$$\text{Electron density } n_e ; 3.5 \times 10^{18} \text{ m}^{-3}.$$

Power flow density (S_{av}) scales with distance of propogation z as

$$S_{av} = S_0 \, e^{-2k_i z} \;\; k_i \;\; \frac{2\nu}{c} \frac{\omega_p^2 / \omega^2}{1 - \omega_p^2 / \omega^2 \;\; ^{1/2}}$$

$$\text{power loss in db} = 10 \log_{10} \frac{S_0}{S_{av}} = 20 k_i z / 2.3$$

$$k_i = \frac{5/9}{2/3} \frac{\nu}{2c} = \frac{3 \times 2.3}{20}$$

$$\nu = 5 \times 10^7 \text{ S}^{-1}$$

Lecture-9

Problem 1: An electromagnetic wave of frequency ώ/2π = 5 MHZ is incident on the ionosphere at angle of incidence $\theta_i = 60^0$.Plasma frequency varies with height h as $\omega_p = \omega_{po}$ exp(h/Ln) with ώpo=2X 10^6 rad/s,Ln=10 km and h measures from the bottom of the ionosphere. Estimate the height of reflection.

Solution

The wave suffers reflection from the height h_T where

ω_p = ώ cosθi = $ώ_{po}$ exp(h_T/L_n)

$h_T = L_n$ ln (ώ/2ώpo) ≈ 20 km

Problem 2: An electromagnetic pulse of carrier frequency ώ impinges normally on a plasma at z=0 .The plasma frequency varies with z as $\omega_p^2 = \omega_{po}^2$ z/L_n. Obtain the group delay time $2\int v_g^{-1} dz$, the time pulse takes to return back to z=0. Here $v_g = c(1 - \omega_p^2/\omega^2)^{1/2}$.

Solution:

The pulse would return from z=z_T where $\omega_p = \omega$,

$$z_T = \frac{\omega^2}{\omega_{po}^2} L_n$$

$$t_g = \frac{2}{c}\int_0^{z_T} \frac{dz}{\left(1 - \frac{\omega_{po}^2}{\omega^2}\frac{z}{Ln}\right)^{1/2}}$$

$$= 4\frac{\omega^2 Ln}{\omega_{po}^2 c}.$$

Problem-3 A laser of frequency ω propagates along $\hat{z}$ in a plasma (z>0)with $\omega_p^2 = \omega^2$ z/Ln, $\nu = \nu_o \frac{z}{L_n}$. At z=0 , $\vec{E} = \hat{x}Ao \ e^{-i\omega t}$.Obtain the laser amplitude in the WkB approximation at z/Ln=0.95 when $\nu_o/w = 10^{-3}$,$\omega L_n/c = 10^3$

Solution

Laser amplitude goes as

$$A = \frac{Ao}{(ck/\omega)^{1/2}} e^{-\int k_i dz}$$

$$k = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2} + \frac{i\nu}{\omega}\frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

$$k_i = \frac{\nu}{2c}\frac{\omega_p^2/\omega^2}{(1-\omega_p^2/\omega^2)^{1/2}}$$

$$\int k_i dz = \frac{\nu_o Ln}{2c}\int_0^{0.95}\frac{\xi^2 d\xi}{(1-\xi)^{1/2}} ; 0.3\nu_o Ln/c = 0.3$$

$$A\Big|_{z=.95L_n} = \frac{A_O}{(.05)^{1/4}} e^{-0.3} \ 1.5A_0.$$

Lecture 10

Problem 1: A plasma wave causes 10% density perturbation. What would be the ratio of electron drift velocity to phase velocity of the wave?

Solution: Electron density perturbation n_1 is related to drift velocity $\vec{v_1}$ through the equation of continuity.

$$\frac{\partial n_1}{\partial t} + \nabla.(n_o \vec{v_1}) = 0$$

$$n_1 = \frac{\vec{k}.\vec{v_1}\, n_o}{\omega}.$$

$$\text{Since } = \vec{v_1} \| \vec{k},\ \frac{v_1}{\omega/k} = \frac{n_1}{n_0} = 0.1$$

The ratio of electron drift velocity to phase of the wave is 0.1

Problem:2 obtain the amplitude of a Langmuir wave electric field of frequency $\omega = 1.1\omega_p$

in a plasma at which the electron drift velocity equals one tenth of electron thermal speed.

$$\text{Let } \vec{E} = \hat{z}\, A\, e^{-(\omega t - kz)}$$

$$\omega^2 = \omega_p^2 + k^2 v_{th}^2 \quad = 1.21\, \omega_p^2 .$$

From the equations of motion and continuity velocity and density perturbations are

$$-i\omega\vec{v_1} = -\frac{e\vec{E}}{m} - v_{th}^2 \quad ikn_1/n_0$$

$$n_1 = n_0\, \vec{k}.\vec{v_1}/\omega$$

$$\vec{v_1} = -\frac{ie\vec{E}\,\omega}{m\left(\omega^2 - k^2 v_{th}^2\right)}$$

$$|v_1| = \frac{v_{th}}{10} \text{ gives}$$

$$|E| = \omega_p^2 m \frac{v_{th}}{10\, e\omega} = mv_{th}\, \omega_p/11e .$$

Problem 3: obtain the dispersion relation for ion acoustic wave in a 50:50 D-T plasma with cold ions and electrons at temperature T_e.

Solution: Let $\emptyset = Ae^{-i(\omega t = kz)}$

Electron density perturbation (adiabatic response).

$n_1 = n_0 \frac{e\emptyset}{Te},$

where n_o is the equilibrium electron density Deuterium- and tritium ion density perturbations are

$$n_{1D} = \frac{n_0}{2} \frac{e\emptyset k^2}{m_D \omega^2},$$

$$n_{1T} = \frac{n_0}{2} \frac{e\emptyset k^2}{3m_D \omega^2/2},$$

where $\mathbf{m_D}$ is the deuterium mass. Using these in the Poisson's equation.

$$\nabla^2\emptyset = \frac{e}{\epsilon_0}(n_1 - n_D - n_{1T}),$$

$$1 + \frac{\omega_p^2}{k^2 v_{th}^2} - \frac{\omega_{pi}^2}{\omega^2} = 0 ,$$

where $\omega_{pi} = (5n_o e^2/6m_D \in_o)^{1/2}, \omega_p = (n_o e^2/m\epsilon_o)^{1/2}$

$$\omega^2 = \frac{k^2 c_s^2}{1 + k^2 c_s^2/\omega_{pi}^2}, c_s = \left(\frac{5T_e}{6m_D}\right)^{1/2}$$

Lecture 11

Problem 1: Estimate the intensity of ion acoustic wave of frequency $\omega = \omega_{pi}/3$ and amplitude (of the electric field) 1V/m in a hydrogen plasma with T_e=10 eV.

Solution:

$$\text{Sav} = \frac{1}{2}\ \omega \frac{\partial \epsilon}{\partial \omega}\ |E^2| v_g$$

$$\epsilon = 1 + \frac{\omega_{pi}^2}{k^2 c_s^2} - \frac{\omega_{pi}^2}{\omega^2}$$

$$\frac{\partial \epsilon}{\partial \omega} = \frac{2}{\omega} \frac{\omega_{pi}^2}{\omega^2}$$

$$V_g = \frac{\partial \omega}{\partial k} = \frac{\omega^3}{k^3 c_s^2} = c_s\left(1 - \omega^2/\omega_{pi}^2\right)^{3/2}$$

$$S_{av} = \frac{\omega_{pi}^2}{\omega^2}\ |E|^2\ c_s\ \left(1 - \omega^2/\omega_{pi}^2\right)^{3/2}$$

$$\cong 2.7\ * 10^5\ \text{W/m}^2$$

Lecture 12

Problem 1: Estimate the energy of an electron beam that excites a co-moving Langmuir wave of frequency $\omega = 1.05\ \omega_p$ in a plasma of electron temperature 1 KeV.

Solution: Langmuir wave dispersion relation is

$$\omega^2 = \omega_p^2 + k^2 v_{th}^2$$

For the unstable wave

$$k=\omega/v_{ob}$$

Thus

$$\frac{\omega^2-\omega_p^2}{\omega^2} = \frac{v_{th}^2}{v_{ob}^2}.$$

$$\text{Beam energy } \epsilon_b = \frac{1}{2} m v_{ob}^2 = \frac{T_e}{1-\omega_p^2/\omega^2} = 11\text{KeV}.$$

Problem2: An $\propto$ particle beam of density $n_{o\alpha}$ and velocity $v_{o\alpha}\hat{z}$ propagates through a plasma of electron density $n_o \gg n_{o\alpha}$ and electron temperature T_e. Ions are cold and singly ionized .Obtain the dispersion relation for the ion acoustic wave propagating at an angle θ to the beam. Deduce the growth rate.

Solution:

Let $\emptyset = Ae^{-i(\omega t-k_x x-k_z z)}$.

Writing $\propto$ particle velocity and density as $\overrightarrow{V_\propto} = \overrightarrow{V_{o\propto}} + \overrightarrow{V_{1\propto}}$, $n_\propto = n_{o\propto} + n_{1\propto}$ and solving the linearnized equations of motion and continuity. we obtain

$$\overrightarrow{V_{1\propto}} = \frac{2e\vec{k}\emptyset}{m_\propto(\omega-k_z V_{o\propto})},$$

$$n_{1\propto} = \frac{2ek^2 n_{o\propto}\varphi}{m_\propto(\omega - k_z v_{o\propto})^2} \ ,$$

where 2e, $m_\propto$ are $\propto$- particle charge and mass.For ions of mass m_i the density perturbation is similarly

$$n_{1i} = \frac{n_o e k^2 \varphi}{m_\propto \omega^2}$$

Electron response is adiabatic with density perturbation.

$$n_1 = n_o e\emptyset/Te$$

Using these in the Poisson's equation $\nabla^2\emptyset = \frac{e}{\epsilon_o}(n_i - n_{1i} - 2n_{1\propto})$ we obtain

$$1+\frac{\omega_{pi}^2}{k^2 c_s^2} - \frac{\omega_{pi}^2}{\omega^2} = \frac{\omega_{p\alpha}^2}{(\omega - k_z V_{o\propto})^2}$$

$$(\omega^2 - \omega'^2)(\omega - K_z V_{o\alpha})^2 = R = \omega^2\omega_{p\alpha}^2/\left(1 + \omega_{p\alpha}^2/k^2 c_s^2\right)$$

$$\text{where } \omega'^2 = \frac{k^2 C^2 s}{\left(1+\frac{k^2 c_s^2}{\omega_{pi}^2}\right)}, \omega_{pi} = (n_o e^2/m_i \epsilon_o)^{1/2}$$

$$\omega_{p\alpha} = \left(4_{n_{o\alpha}} e^2/m_\alpha \in_o\right)^{1/2}, c_s = (T_e/m_i)^{1/2}$$

Most unstable mode occurs when $k_z V_{o\alpha} = \omega'$

$$V_{o\alpha} \cos\theta = c_s/\left(1 + k^2 c_s^2/\omega_{pi}^2\right)^{1/2}$$

Writing $\omega = \omega' + \delta \quad = k_z V_{o\alpha} + \delta$ we obtain

$$\delta^3 = \left(\frac{R}{2\omega'}\right) e^{2il\pi}, l = 0,1,2$$

$$\text{Growth rate } r = I_m \delta = \frac{\sqrt{3}}{2}\left(\frac{R}{2\omega'}\right)^{1/3}.$$

The unstable ion acoustic wave has frequency $\omega' = \omega_{pi} \cdot \left(1 - v_{o\alpha}^2 \cos^2\theta / c_s^2\right)^{1/2}$ which Increases with increasing θ.

Problem:3 A non-relativistic electron beam of density n_{ob} and velocity $V_{ob.}\hat{z}$ propagates through a plasma of electron density n_o having static magnetic field $B_s\hat{z}$.Perturb the equilibrium by an electrostatic wave $\emptyset = A\exp[-i(\omega t - k_X x - k_z z)]$.Deduce the dispersion relation ignoring collisions ,thermal effects and ion motion.

Solution: Writing beam velocity $\overrightarrow{v_b} = v_{ob}.\hat{z} + \overrightarrow{v_{1b}}$ and linearizing the equation of motion,

We obtain

$$-i\,(\omega - k_z v_{ob})\overrightarrow{v_{1b}} + \overrightarrow{v_{1b}}\, x\, \overrightarrow{\omega_c} = \frac{ei\vec{k}}{m}\varphi,$$

$$v_{1bx} = -\frac{ek_x\emptyset(\omega - k_z v_{ob})}{m\left[(\omega - k_z v_{ob})^2 - \omega_c^2\right]}$$

$$V_{1bz} = -\frac{ek_x\emptyset}{m(\omega - k_z v_{ob})}$$

Using beam density $n_b = n_{ob} + n_{1b}$ in the linearized equation of continuity. we get

$$n_{1b} = n_{ob}\frac{\vec{k}\,.\overrightarrow{v_{1b}}}{\omega - k_z\, v_{ob}} = -\frac{n_{ob}e\varphi}{m}\left[\frac{k_x^2}{(\omega - k_z v_{ob})^2 - \omega_c^2} + \frac{k_z^2}{(\omega - k_z v_{ob})^2}\right]$$

The plasma electron density perturbation n_1, can be recovered from n_{1b} by putting$V_{ob} = 0$,$n_{ob} \rightarrow n_o$,

$$n_1 = -\frac{n_o e\emptyset}{m}\left[\frac{k_x^2}{\omega^2 - \omega_c^2} + \frac{k_z^2}{\omega^2}\right]$$

Poisson's equation: $\nabla^2\emptyset = \frac{e}{\epsilon_o}(n_1 + n_{1b})$ gives the dispersion relation,

$$1- \frac{\omega_p^2\, k_x^2/k^2}{\omega^2-\omega_c^2} - \frac{\omega_p^2}{\omega^2}\frac{k_z^2}{k^2} = \frac{\omega_{pb}^2\, k_z^2/k^2}{(\omega-k_z\, v_{ob})^2} + \frac{\omega_{pb}^2\, k_x^2/k^2}{(\omega-k_z\, v_{ob})^2-\omega_c^2}$$

The beam can drive the wave unstable when

(i) $\omega - k_z\, v_{ob} \cong 0$ (Cerenkov resonance),

(ii) $(\omega - k_z\, v_{ob})^2 - \omega_c^2 \cong 0$ (cyclotron resonance).

Lecturer 16

Problem 1: A relativistic electron beam of density n_{ob} ,charge –e , rest mass m and velocity

$v_o\hat{z}$ propagates through a wiggler $\overrightarrow{B_w} = B_0\,(\hat{x} - i\hat{y})\,e^{ik_w z}$ in the presence of a guide magnetic field $B_s\hat{z}$ .Obtain the perturbed electron velocity.

Solution: Electron energy $\gamma mc^2$is constant in a dc magnetic field.Writing beam velocity

$\vec{v} = v_o\hat{z} + \overrightarrow{v_w}$ the linearized equation of motion takes the from

$$m\gamma\left(\frac{\partial\overrightarrow{v_w}}{\partial t} + v_0\frac{\partial}{\partial z}\overrightarrow{v_w}\right) = -e\overrightarrow{v_0} * \overrightarrow{B_w} - e\,\overrightarrow{v_w}\ x\ B_s\hat{z}\,,$$

Defining $\omega_c = eBs/m$ and taking $\overrightarrow{v_1} = \vec{a}\exp(ik_w z)$ i.e , $\frac{\partial}{\partial t}$=0 and $\frac{\partial}{\partial z}$ = ik_w we obtain

$$ik_w\,v_0\overrightarrow{v_w} + \overrightarrow{v_w}\ x\ \hat{z}\frac{\omega_c}{\gamma} = -\frac{iev_0\overrightarrow{B_w}}{m\gamma},$$

$$\overrightarrow{V_w} = -\frac{ev_0.\overrightarrow{B_w}}{m\gamma(k_w v_0 - \omega_c/\gamma)}\ .$$

It shows cyclotron resonance enhancement when $k_w v_0 = \omega_c/\gamma$.

Problem:2 In a plasma wave wiggler free electron laser one passes a relativistic electron beam of velocity $V_{0b}\hat{z}$ through a plasma that has a plasma wave of frequency ω_0 and wave number $\overrightarrow{k_0} = k_{o\perp}\hat{x} - k_{o||}\hat{z}$.As the beam emits radiation of frequency ω and wave number $k\hat{z}$ it excites a space charge mode of frequency $\omega + \omega_0$,and wave vector $\vec{k} + \overrightarrow{k_0}$ that is in Cerenkov resonance with the beam . Deduce the frequency of radiation.

Solution :

Cerenkov resonance occurs when

$$\omega - \omega_0 = \left(\vec{k} - \overrightarrow{k_0}\right).\overrightarrow{v_{0b}} = (k + k_{011})v_{0b}$$

Further, $\omega^2 = \omega_p^2 + k^2c^2$ in a plasma of plasma frequency ω_p when beam density is low.Using $k = \left(\omega^2 - \omega_p^2\right)^{1/2}/c$ in the above equation one obtains.

$$\omega = \gamma_o^2\left[k_{o11}v_{ob} + \omega_0 + (v_{0b/c})\left\{\left(k_{0||}v_{ob} + \omega_0\right)^2 - \frac{\omega_p^2}{\gamma_o^2}\right\}^{1/2}\right],$$

where $\gamma_0 = \left(1 - v_{ob}^2/c^2\right)^{1/2}$.The kinetics of the process demands that the plasma wave posses a component of electric field parallel to the electric field of radiation mode , hence it must move at an angle to $\overrightarrow{v_{ob}}$ .In case $k_{0||} = 0$, $\gamma_0 \gg 1$,$\omega \cong 2\gamma_o^2\,\omega_0$.

Problem 3: If one introduces plasma of plasma frequency ω_p in a free laser of wriggler wave number $k_w\hat{z}$ (to aid the propagation of the electron beam of velocity $V_{ob}\hat{z}$). What would be the operating frequency of FEL?

Solution:

$$\omega = (k + k_w)v_{ob}$$

$$\omega^2 = \omega_p^2 + k^2c^2$$

These equations give the operating frequency

$$\omega = \gamma^2\left[k_w v_{ob} + v_{ob}/c\left(k_w^2 v_{ob}^2 - \frac{\omega_p^2}{\gamma_o^2}\right)^{1/2}\right]$$

The frequency of FEL decreases marginally with ω_p .

Problem 4: A slow TM mode of dielectric lined wave guide of frequency ω_0 and wave number k_{0z} acts as a wiggler when an electron beam of velocity $v_{ob}\hat{z}$ propagates through it. Obtain the frequency ω of the generated radiation.Take the wave vector of radiation $k_z = (\omega/c)$.

Solution:

Phase synchronism condition

$$\omega - \omega_o = (k - k_{oz})v_{ob}$$

$$\text{gives } \omega = \frac{\omega_0 - k_{0z}V_{0b}}{1 - V_{ob}/c} \cong 2\gamma_o^2(\omega_0 - k_{oz}v_{0b}) \text{ for } \gamma_o^2 \gg 1.$$

For a counter propagating TM mode k_{0z} is negative hence ω is much larger. One may employ a counter propagating laser of wavelength λ_L as wriggler then the generated radiation wavelength $\lambda = \lambda_L/4\gamma_o^2$ may lie in the X-ray region. This is a viable scheme for coherent X-ray laser.

Problem 5: A plasma of plasma frequency ω_p is placed in a strong magnetic field $B_s\hat{z}$(with electron cyclotron frequency $\omega_c \gg \omega_p$). An electron beam of density n_{0b} and velocity $v_{ob}\hat{z}$ propagates through it along the magnetic field. Obtain the frequency of radiation it would excite.

Solution:

The finite components of permittivity tensor of the plasma (at frequency ω) with magnetic field along $\hat{z}$ and $\omega_p/\omega_c, \omega/\omega_c \ll 1$ would be $\epsilon_{xx} = \epsilon_{yy} = 1, \epsilon_{zz} = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2/\gamma_o^2}{(\omega - k_z V_{ob})^2}$

$$\text{where } \gamma_o = \left(1 - v_{ob}^2/c^2\right)^{-1/2}, \omega_{pb} = (n_{ob}e^2/m\,\epsilon_o)^{1/2}$$

For an electromagnetic wave $\vec{E} = \vec{A}e^{-i(\omega t - k_{xx} - R_z z)}$, Maxwell's equations $\nabla \times \vec{E} = i\omega\mu_o\vec{H}$ and $\nabla \times \vec{H} = i\omega\,\epsilon_o\underline{\underline{\epsilon}}.\vec{E}$, on taking curl of the former and using the latter along with replacing ∇ by i $\vec{k}$, lead to

$$\underline{\underline{D}} \cdot \vec{E} = \left[k^2 I - \vec{k}\vec{k} - \frac{\omega^2}{c^2}\underline{\underline{\epsilon}}\right].\vec{E} = 0$$

giving the dispersion relation $\left|\underline{\underline{D}}\right| = 0$, or

$$k_x^2 + \left(k_z^2 - \frac{\omega^2}{c^2}\right)\left(1 - \frac{\omega_p^2}{\omega^2}\right) = \frac{\omega_{pb}^2(k_z^2 - \omega^2/c^2)}{(\omega - k_z V_{ob})^2 \gamma_o^3}$$

The instability occurs at a frequency at which the factor on the left hand side is zero with $\omega = k_z v_{ob}$. This gives

$\omega = \left(\omega_p^2 - k_x^2 v_{ob}^2\, \gamma_o^2\right)^{1/2}$.

In fact the dispersion relation, when beam is not there, gives two roots one with phase velocity greater than c and the other less than c. The letter mode is excited by the beam.

Lecturer 18

Problem 1: Two counter streaming relativistic electron beams of density n_{ob} each and drifts $v_{ob}\hat{z}$, $-v_{ob}\hat{z}$ propagate in a cold collisionless plasma of density n_o. Perturb the equilibrium by an electromagnetic perturbation $\vec{E} = \hat{z}A.\exp[-i(\omega t - kx)]$. Obtain the growth rate.

Solution:

Let the velocity of the first beam be

$$\overrightarrow{v_b} = V_{ob}\hat{z} + \overrightarrow{v_{1b}}$$

The gamma factor $\gamma = \left(1 - v_b^2/c^2\right)^{-1/2}$ on linearization can be written as

$$\gamma = \gamma_o + \gamma_0^3 V_{1bz} V_{ob}/c^2,$$

$$\gamma\overrightarrow{V_b} = \gamma_o V_{ob}\hat{z} + \gamma_o V_{1bx}\hat{x} + \gamma_o^3 V_{1bz}\hat{z}$$

The lineralized equation of motion, on writing wave magnetic field $\vec{B} = \vec{k} \times \vec{E}/\omega$, gives

$$\left(\frac{\partial}{\partial t} + v_{ob}\frac{\partial}{\partial z}\right)(\gamma_o v_{1bx}\hat{x} + \gamma_o^3 v_{1bz}\hat{z})$$

$$= -\frac{e}{m}\vec{E} - \frac{eV_{ob}}{m\omega}\hat{z} \times \left(\vec{k} \times \vec{E}\right)$$

$$v_{1bz} = \frac{eE_z}{mi\,\omega\,\gamma_0^3}, V_{1bx} = \frac{ekv_{ob}}{mi\,\omega^2\,\gamma_0}E_z$$

The density perturbation is

$$n_{1b} = \frac{n_{ob}}{\omega}k\,v_{1bx} = \frac{n_{ob}e\,k^2 v_{ob}}{mi\omega^3\gamma_o}E_z$$

For the second beam velocity and density perturbations are

$$v_{2bz} = v_{1bz}, v_{2bx} = -v_{1bx}, n_{2b} = -n_{1b}$$

For plasma electrons perturbed density is zero and perturbed velocity is

$$\vec{v} = \frac{e\vec{E}}{mi\omega}$$

The current density at $\omega, \vec{k}$ is

$$\vec{J} = -e n_{ob}\, \overrightarrow{v_{1b}} - n_{ob} e \overrightarrow{v_{2b}} - n_{1b} e v_{ob} \hat{z} + n_{2b} e v_{ob} \hat{z} - n_o e \vec{v}$$

$$= -\hat{z} \frac{2 n_{ob} e^2}{mi\omega\gamma_o^3} \left(1 + \gamma_o^2 \frac{k^2 . V_{ob}^2}{\omega^2}\right) E_z - \frac{n_o e^2}{mi\omega}\, E_z \hat{z}\,.$$

Using this in the wave equation

$$\nabla^2 \vec{E} - \nabla(\nabla . \vec{E}) + \frac{\omega^2}{c^2} \vec{E} = -\frac{i\omega}{\epsilon_o\, c^2} \vec{J}\,,$$

we obtain

$$\omega^2 = \frac{\omega_{pb}^2\, k^2\, v_{ob}^2/\gamma_o}{k^2 c^2 + \omega_p^2 + \omega_{pb}^2/\gamma_o^3}\,,$$

where $\omega_{pb}^2 = \frac{2 n_{ob} e^2}{m\epsilon_o}, \omega_p^2 = n_o e^2/m\epsilon_o$.

Lecture 19

Problem 1: A collisionless cold plasma in equilibrium has a dc magnetic field $B_s\hat{z}$ a dc electric field $E_o\hat{x}$, gravity -g$\hat{x}$ and a density gradient $\nabla n_o \approx \hat{x}\, n_o 1/Ln$. It is perturbed by an electrostatic perturbation $\emptyset = A\exp[-i(\omega t - k_y y)]$ with $\omega \ll \omega_{ci}, \omega_{pi}$, the ion cyclotron and ion plasma frequencies. Obtain the growth rate (of the modified Rayleigh Taylor instability).

Solution:

The dc drift of electrons and ions are

$$\vec{v_o} = \frac{\vec{E_o} * \vec{Bs}}{B_s^2} = v_o\hat{y}, v_o = -E_o/Bs$$

$$\vec{v_{oi}} = (v_o + v_{og})\hat{y} \;\; v_{og} = \frac{g}{\omega_{ci}},$$

Linearized equations of motion and continuity for ions of mass m_i and charge +e, give

$$\frac{\partial \vec{v_{1i}}}{\partial t} + (v_0 + v_g)\frac{\partial}{\partial y}\vec{v_{1i}} - \vec{v_{1i}} \times \hat{z}\omega_{ci} = -\frac{i\, e\, \vec{k}\, \varphi}{m_i},$$

$$\frac{\partial n_1}{\partial t} + \nabla.(n_o\vec{v_{1i}} + n_1(v_o + V_{og})\hat{y}) = 0,$$

$$V_{1iy} = -\frac{ek\emptyset}{m_i\omega_{ci}^2}(\omega - kv_o - kv_{og}),$$

$$n_{1i} = -\frac{n_0 e k^2 \emptyset}{m_i \omega_{ci}^2} - \frac{n_0 e\, k\, \emptyset}{L_n m_i \omega_{ci}(\omega - k\, v_0 - k v_{0g})}.$$

Similarly the electron density perturbation is

$$n_1 \cong -\frac{n_0 e k \emptyset}{L_n m_i \omega_{ci}(\omega - k\, v_0)}.$$

Quarneutrnlity, $n_1 \cong n_{1i}$, gives the dispersion relation,

$$\frac{kL_n}{\omega_{ci}} = \frac{1}{\omega - k\,v_0} - \frac{1}{\omega - k\,v_0 - k\,v_{0g}}$$

$$\omega \simeq k\,v_0 + i\sqrt{g/L_n}.$$

The elective field introduces a real part in the frequency. In a frame moving with velocity $\vec{v}_0$, it is a purely growing instability.

Lecture 20

Problem 1: A magnetron has coaxial inner and outer cylinders of radii a and b and a static magnetic field $\vec{B}_S$ along the axis $\hat{Z}$. The inner cylinder is given a negative potential V_0 while the outer one is grounded. Obtain the maximum value of B_S. So that the electrons released at the cathode, with velocity 0, do not reach the anode.

Solution:

The radial and azimuthal equations of motion for an electron are:

$$\ddot{r} - r\,\dot{\emptyset}^2 = -\frac{eE_r}{m} - r\dot{\emptyset}\,\omega_c\,,$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\emptyset}\right) = \dot{r}\,\omega_c.$$

The last equation (with initial condition at t = 0, $\dot{\emptyset} = 0$) gives

$$\dot{\emptyset} = \frac{\omega_c}{2}(1 - a^2/r^2).$$

When the electron just turns around at r = b, at this point $\dot{r} = 0$, $\vec{v} = b\dot{\emptyset}\hat{\emptyset}$. The kinetic energy gained by the electron on reaching the anode is

$$\frac{1}{2}\,m\,v_{\emptyset}^2 = eV_0 \;\; or \;\; v_{\emptyset} = b\,\dot{\emptyset} = \left(\frac{2eV_0}{m}\right)^{1/2}.$$

This gives the requisite ω_c from the earlier equation

$$\omega_c = \frac{eB_s}{m} = \frac{2}{(1 - a^2/b^2)b}\left(\frac{2eV_0}{m}\right)^{^/2}.$$

Problem 2: A cylindrical ion channel of radius a has singly ionized ions of density n_0. A non-relatistic electron is released at (r_0, 0, 0) with $\nu_o\,\hat{z}$ velocity. It executes bounce motion. Obtain its bounce frequency.

Solution : The electric field in the ion channel, on using Gauss`s law, is

$E_r = \frac{n_0 e r}{2\epsilon_0}$.

Under this field, and with initial velocity along $\widehat{\boldsymbol{Z}}$**,** the electron motion is confirmed in the x-z plane. The x-component of equation of motion is

$$m\frac{d\,\nu_x}{dt} = -\frac{n_0\,e^2}{2\,\epsilon_0}x$$

Multiplying both sides by ν_x and integrating over time, using the initial conditions t = 0, ν_x = 0, x = r_o, we obtain

$$\frac{1}{2}\,m\nu_x^2 = \frac{n_0 e^2}{4\epsilon_0}\,(r_0^2 - x^2)$$

$$\int_{r_0}^{x}\frac{dx}{(r_0^2 - x^2)^{1/2}} = \frac{\omega_P t}{\sqrt{2}};\ \omega_P = \left(\frac{n_0 e^2}{m\epsilon_0}\right)^{1/2},$$

x = r_o Sin (ω_P t / √2 + π/2).

The bounce frequency is $\omega_b = \omega_P / \sqrt{2}$. The z component of velocity remains constant.

Problem 3 : Show that for an electron of change –e, mass m and velocity $\vec{\nu}$, moving in a magnetic field $\vec{B}_s = B_0^{`}\, r\,\widehat{\emptyset}$ (where $B_0^{`}$ is a constant and r, ϕ refer to cylindrical polar coordinate system), $m\nu_z$ + e B_o' r^2/2 is a constant of motion.

Solution:

$$m\frac{d\,\nu_z}{dt} = -e\left(\vec{\nu} * \vec{B}\right)_z = -e\,B_0^{`}\,\nu_r\,r = -\frac{d}{dt}\left(\frac{e\,B_o^{'}\,r^2}{2}\right).$$

Thus $m\nu_z + eB_0^{`}r^2/2$ = constant of motion

Lecture 21

Problem 1: A long wire aligned along $\hat{z}$ carries current I = 10 KA. An electron is released at distance r = 0.1 m, from the wire with velocity $\vec{v}_0 = (10^7\ m/s)\hat{z}$. Obtain the ∇B and curvature drifts of the electron.

Solution: The magnetic field of a long wire, on using Ampere's law is

$$\vec{B} = \hat{\emptyset}\frac{\mu_0 I}{2\pi r}.$$

Electron gyrates about the field line with transverse speed $v_\perp = v_0$. The radius of curvature of the magnetic field is r and ∇B = $\nabla B = -\frac{\mu_0 I}{2\pi r^2}\ \hat{r}$ .Thus

$$\vec{v}_{\nabla B} = +\frac{v_\perp^2}{2\omega_c B^2}\ \nabla B \times \vec{B}$$

$$= -\frac{v_0^2}{2\omega_c r}\ \hat{z} = -1.6 \times 10^5\ \frac{m}{s}\ \hat{z}$$

$$\vec{v}_{curv} = 0\ as\ v_\emptyset = 0.$$

Lecture 22

Problem 1: Estimate the loss cone angle of a mirror machine having magnetic coils of radius R separated by distance L = 6R.

Solution: Treating origin on the axis and midway between the coils, the magnetic field on the axis at axial position z, due to the coils each having N turns carrying current I, is

$$B_z = \frac{\mu_0 I R^2 N}{2}\left[\left(R^2 + \left(\frac{L}{2} + z\right)^2\right)^{-3/2} + \left(R^2 + \left(\frac{L}{2} - z\right)^2\right)^{-3/2}\right].$$

At z = 0, B = B_{min} = (μ_0 I N) / (R $10^{3/2}$).

At z = L/2, B = B_{max} = M_0 I N (1 + $37^{-3/2}$) / 2R.

Loss cone angle = $\theta_c \cong \sin^{-1} (1/15)^{1/2}$.

Problem 2: A mirror machine has field coils, each of radius R, turns N and current I, placed at z = ± L/2, where L = 4R. An electron near the mirror midpoint has velocity $\vec{\nu} = \nu_0 \cos\theta_0 \hat{z} + \nu_0 \sin\theta_0 \hat{\emptyset}$ $with\ \theta_0$ close to π/2. Estimate the bounce frequency and amplitude of oscillation.

Solution: The magnetic field on the mirror axis is

$$B_z = \frac{\mu_0 I\ N\ R}{2}\left[\left(R^2 + \left(z - \frac{L}{2}\right)^2\right)^{-3/2} + \left(R^2 + \left(z + \frac{L}{2}\right)^2\right)^{-3/2}\right]$$

$$\simeq \frac{\mu_0\ I\ N}{R\ 5^{3/2}}\left(1 + \frac{6}{5}\frac{z^2}{R^2}\right) = B_z\ (o)\ \left(1 + \frac{6}{5}\frac{z^2}{R^2}\right),$$

$$m\frac{d^2z}{dt^2} = -\mu\frac{\partial B_z}{\partial z} = -\frac{m\ \nu_0^2 cos^2\theta_0}{2\ B_z(o)}\ \ \frac{12}{5}\ B_z(o)\frac{z}{R^2},$$

$$\frac{d^2z}{dt^2} = -\ \omega_b^2 z.$$

Bounce frequency $\omega_b = \sqrt{\frac{6}{5}}\ \frac{\nu_0}{R}\cos\theta_0$

At the maximum axial displacement z = z_A.

$$\frac{1}{2} m v_0^2 sin^2\theta_0 \,/\, B_z(o) = \frac{1}{2} m v_0^2 \,/\, B_z(z_A)$$

$$\frac{B_z(z_A)}{B_z(o)} = 1 + \frac{6}{5}\,\frac{z_A^2}{R^2} = \frac{1}{sin^2\theta_0}$$

$$z_A = \sqrt{5/6}\, R \cot\theta_0.$$

Lecture 25

Problem 1: A plasma cylinder of radius a, having axis along $\hat{z}$, has equilibrium current density $\vec{J} = \hat{z} J_0 (1 - r^2/a^2)$. Deduce the magnetic field and plasma pressure profiles.

Solution :

$$\vec{J} = \hat{z} J_0 (1 - r^2/a^2).$$

$$\nabla \times \vec{B} = \mu_0 \vec{J},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = \mu_0 J_0 (1 - r^2/a^2).$$

$$B_\phi = \frac{\mu_0 J_0}{2} r(1 - r^2/2a^2).$$

In equilibrium

$$\nabla P = \vec{J} \times \vec{B},$$

$$\frac{\partial P}{\partial r} = -J_0\left(1 - \frac{r^2}{a^2}\right)\frac{\mu_0 J_0}{2} r\left(1 - \frac{r^2}{2a^2}\right),$$

$$P = P_0 - \frac{\mu_0 J_0^2}{4} r^2\left(1 - \frac{3}{4}\frac{r^2}{a^2} + \frac{r^4}{6\,a^4}\right).$$

Problem 2: A magnetic cusp comprises two closely spaced parallel field coils (with axes $\parallel \hat{z}$) carrying currents in opposite directions. Consequently the axial magnetic field on the left ($z < 0$) is opposite to the field on the right ($z > 0$). In the cusp region axial field changes rapidly from –ve to +ve value, hence a strong radial field B_r exists there. The vector potential for such a system has finite ϕ component A_ϕ (r, z). Show that for an electron approaching the cusp from left, $r(p_\phi + eA_\phi)$ is a constant of motion, where p_ϕ is the ϕ component of momentum.

Solution:

$$\vec{B} = \nabla \times \vec{A}.$$

Azimuthal equation of motion is

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\emptyset}\right) = -\frac{e}{m}(\vec{\nu} \times \vec{B})_{\emptyset}$$

$$= -\frac{e}{m}\left[\nu_z\left(-\frac{\partial A_{\emptyset}}{\partial z}\right) - \frac{\nu_r}{r}\frac{\partial(rA_{\emptyset})}{\partial r}\right]$$

$$= \frac{e}{mr}\frac{d(rA_{\emptyset})}{dt}.$$

Thus gives r (p_ϕ - e A_ϕ) = constant.

Lecture 28/29

Problem 1: An electromagnetic wave of frequency ω propagates along the direction of static magnetic field $B_S\hat{z}$ in a collision less plasma of $\omega_P = \omega/2$, $\omega_c = 2\omega/3$. At z =0, $\vec{E}(0,t) = A\,(\hat{x} + 2\,i\,\hat{y})\exp(-\,i\,\omega\,t). Write\ \vec{E}\,(z,t)$.

Solution:

Express $\overrightarrow{E}(z,t)$ as sum of RCP and LCP:

$$\overrightarrow{E}(z,t) = A_R(\hat{x} + i\,\hat{y})\,e^{-i(\omega t - k_R z)} + A_L(\hat{x} - i\,\hat{y})\,e^{-i(\omega t - k_L z)}$$

$$\overrightarrow{E}(o,t) = [(A_R + A_L)\hat{x} + i(A_R - A_L)\hat{y}\,]e^{-i\omega t} = A\,(\hat{x} + 2\,i\,\hat{y})e^{-i\omega t}$$

Equating the $\hat{x}$ and $\hat{y}$ terms on both sides,

$A_R + A_L = A$,

$A_R - A_L = 2A$,

$A_R = 3A/2$, $A_L = -\,A/2$.

Further, $k_R = \frac{\omega}{c}\left(1 - \frac{\omega_P^2}{\omega(\omega - \omega_c)}\right)^{1/2} = \omega/2c$,

$$k_L = \frac{\omega}{c}\left(1 - \frac{\omega_P^2}{\omega(\omega + \omega_c)}\right)^{1/2} = \sqrt{\frac{17}{20}}\frac{\omega}{c} \simeq 0.93\,\omega/c,$$

$$\overrightarrow{E}(z,t) = \frac{3A}{2}(\hat{x} + i\,\hat{y})\,e^{-i(\omega t - \omega z/2c)} - \frac{A}{2}(\hat{x} - i\hat{y})\,e^{-i(\omega t - 0.93\,\omega z/c)}$$

Lecture 29

Problem 2: A RCP pulse of carrier frequency ω propagates along static magnetic field $B_s\,\hat{z}$ in a collisionless magnetized plasma with $\omega_P = 2\omega$, $\omega_c = 1.1\omega$. At z = 0,

$$\vec{E}\,(0,t) = \hat{x} + i\,\hat{y}\,A\,e^{-t^2/\tau^2}\exp(-i\,\omega\,t).\,Write\ \vec{E}\,(z,t).$$

Solution:

$$\vec{E}(z,t) = (\hat{x} + i\,\hat{y})\,Ae^{-(t\,-\,z/v_g)^2/\tau^2}\ e^{-i(\omega t\,-\,k_R\,z\,)},$$

where

$$k_R = \frac{\omega}{c}\left(1 - \frac{\omega_P^2}{\omega(\omega-\omega_c)}\right)^{1/2} = 6.4\ \omega/c,$$

$$v_g = \frac{\partial\omega}{\partial k_R} = \frac{k_R c^2}{\omega + 0.5\,\omega_P^2/(\omega-\omega_c)^2} = 0.03\,c.$$

Problem 3: Obtain the heating rate of electrons due to 100 GHz, 1 GW/m^2 RCP mode in a plasma when $\omega_c = 0.9\omega$, $\omega_P = 0.3\omega$, $\nu = 10^{-4}\omega$, where ω is the frequency of the mode and ω_p, ω_c and ν are the plasma frequency, cyclotron frequency and collision frequency of electrons.

Solution:

RCP Mode: $\vec{E} = A(\hat{x} + i\,\hat{y})\,e^{-i(\omega t\,-\,k_R z)}$,

$$\vec{H} = k_R\hat{z}\times\vec{E}\,/\mu_o\omega\,,$$

$$k_R \cong \frac{\omega}{c}\left(1 - \frac{\omega_P^2}{\omega(\omega-\omega_c)}\right)^{1/2} = 0.31\,\frac{\omega}{c},$$

$$S_{av} = \frac{1}{2}\,\vec{E}^*\times\vec{H} = \frac{k_R A^2}{\mu_0\omega} = 10^9\ W/m^2$$

$A = 1.1\times10^6$ V/m

Electron drift velocity

$$\vec{v} = \frac{e\,\vec{E}}{mi(\omega - \omega_c + i\,\nu)}$$

Heating rate

$$\vec{H} = \frac{1}{2}\,R_e\left[-e\,\vec{E}^*.\vec{v}\right] = \frac{e^2 A^2 \nu}{m(\omega - \omega_c)^2} = 5 \times 10^{-10} W = 3.1\, G\,e\,V/s.$$

Problem 4: A RCP whistler of intensity 100 MW/m^2 and frequency $\omega/2\pi$ = 1 GHz propagates along a static magnetic field $B_s\,\hat{z}$ in a plasma of plasma frequency $\omega_P = 10^2\omega$, electron cyclotron frequency $\omega_c = 10^2\omega$ and collision frequency $\nu = 10^{-3}\,\omega_c$. Assuming the average energy loss per collision to ions = $3\delta\,(T_e - T_0)/2$ with $\delta = 10^{-4}$ and T_e and T_0 as the temperature of electrons and ions, estimate $T_e - T_0$.

Solution:

Whistler, $\overrightarrow{E} = A(\hat{x} + i\,\hat{y})\,e^{-i(\omega t\,-\,k_R z)}$ imparts oscillatory velocity to electrons,

$$\vec{v} = \frac{e\,\vec{E}}{mi(\omega - \omega_c + i\,\nu)}$$

and heats them at the time average rate

$$H = \frac{1}{2}\,R_e\left[-e\,\vec{E}^*.\vec{v}\right] = \frac{e^2 A^2 \nu}{m(\omega - \omega_c)^2}\,.$$

In the steady state

$$H = \frac{e^2 A^2 \nu}{m(\omega - \omega_c)^2} = \frac{3}{2}\,\delta\nu(T_e - T_0)\,,$$

$$T_e - T_0 = \frac{2e^2 A^2}{3m\delta(\omega - \omega_c)^2}\,.$$

Whistler intensity

$$S_{av} = \frac{1}{2}\left(\vec{E}^{*} \times \vec{H}\right) = \frac{A^2}{\mu_0 \omega} R_e(k_R) = 10^8 \frac{W}{m^2}$$

$$k_R \cong \frac{\omega}{c}\left(1 - \frac{\omega_P^2}{\omega(\omega - \omega_c)}\right)^{1/2} \simeq 10 \frac{\omega}{c}$$

A = 6 × 10^4 V/m, $T_e - T_0 \cong 10$ e V.

Problem 5: A linearly polarized electromagnetic wave, of frequency ω and amplitude A_0 propagating along $\hat{z}$, is normally incident on a collisionless plasma (z > 0) having dc magnetic field $B_s \hat{z}$ and $\omega_p = 2\omega_c = 3\omega$. Estimate the amplitude and polarization of the wave deep inside the plasma.

Solution:

Incident wave $\vec{E} = \hat{x} A_0\, e^{-i(\omega t - \omega z/c)}$ can be written as sum of RCP and LCP

$$\vec{E} = A_R(\hat{x} + i\hat{y})\, e^{-i(\omega t - \omega z/c)} + A_L(\hat{x} - i\hat{y})\, e^{-i(\omega t - \omega z/c)}$$

$A_R = A_L = A_0/2$.

The refractive indices of the plasma for RCP and LCP are

$$\eta_R = \left(1 - \frac{\omega_P^2}{\omega(\omega - \omega_c)}\right)^{1/2} \simeq 4.4,$$

$$\eta_L = \left(1 - \frac{\omega_P^2}{\omega(\omega + \omega_c)}\right)^{1/2} \simeq i\, 1.6.$$

Amplitude transmission coefficients for RCP and LCP are

$T_R = 2 / (1 + \eta_R) \simeq 0.37$,

$T_L = 2 / (1 + \eta_L) = 2 / (1 + 0.16\, i)$.

The LCP field decays in the plasma as exp (-1.6 ωz/c). Deep inside the plasma only RCP will survive and its amplitude would be $T_R A_R = 0.18 A_0$.

Problem 6: An electromagnetic wave of frequency ω is normally incident from vacuum on an unmagnetized collisionless plasma (z > 0). It suffers 100% reflection with phase change of π/2. Estimate ω_P/ω. If one applies a static magnetic field $B_s\,\hat{z}$ to the plasma, normal to the interface with $\omega_c / \omega = 2$. how much phase change on reflection would occur for (i) RCP and (ii) LCP ?

Solution:

Amplitude reflection coefficient

$$R_A = \frac{1-\eta}{1+\eta},\ \eta = (1-\omega_P^2/\omega^2)^{1/2}.$$

100% reflection occurs when η is imaginary (η = iα). Then R_A = exp (-iϕ), where ϕ is the phase change on reflection $\phi = 2\tan^{-1}\alpha = \pi/2 \Rightarrow \alpha = \left(\frac{\omega_P^2}{\omega^2} - 1\right)^{1/2}$; $\omega_P = \sqrt{2}\,\omega$.

For RCP and LCP the amplitude reflection coefficients are

$$R_R = \frac{1-\eta_R}{1+\eta_R}, R_L = \frac{1-\eta_L}{1+\eta_L},\ \eta_{R,L} = \left(1 - \frac{\omega_P^2}{\omega(\omega \mp \omega_c)}\right)^{1/2} = \sqrt{3},\ 1/\sqrt{3}$$

R_R is –ve, hence RCP suffers a phase change of Π. R_L is +ve, hence there is no phase change on reflection for LCP.

Lecture 30

Problem 1: A 20 GHz microwave propagating through a plasma of $\omega_P = 3 \times 10^{10}$ rad/s suffers. Faraday rotation of 45^0, in propagating along dc magnetic field $B_S\,\hat{z}$, a distance of 1 meter. Estimate the magnetic field. You may assume $\omega_c^2 << \omega^2$.

Solution :

$$k_{R,L} = \frac{\omega}{c}\left(1 - \frac{\omega_P^2}{\omega(\omega \mp \omega_c)}\right)^{1/2} \cong \frac{\omega}{c}\left(1 - \frac{\omega_P^2}{2\omega(\omega \mp \omega_c)}\right),$$

Faraday rotation

$$\theta = (k_L - k_R)\, z/2 = \frac{\omega_P^2}{c}\,\frac{\omega_c\, z}{\omega^2 - \omega_c^2} = \pi/4$$

$$\omega_c = \frac{\pi}{4}\,\frac{c\,\omega^2}{z\,\omega_P^2} = 3.7 \times 10^9$$

$$B_S = \omega_c\, m/e = 2.3 \times 10^{-2}\ Tesla.$$

Problem 2: Obtain the time average Pointing’s vector of the x-mode propagating normal to static magnetic field $B_S\,\hat{z}$ in a collisionless plasma.

Solution : For the X mode

$$\vec{E} = A(\hat{y} + \alpha\,\hat{x})\, e^{-i(\omega t - kx)}$$

$$\alpha = -\frac{\in_{xy}}{\in_{xx}} = imaginory$$

$$\vec{H} = \vec{k} \times \vec{E}/\omega\,\mu_0$$

$$\vec{S}_{av} = \frac{1}{2}\mathrm{Re}[\vec{E}^* \times \vec{H}]$$

$$= \frac{1}{2\mu_0\,\omega}\mathrm{Re}[\vec{k}\,\vec{E}.\vec{E}^* - \vec{E}\,k\,E_x^*]$$

$$= \frac{k\,A^2}{2\,\mu_0\,\omega}\ \hat{x}.$$

Lecture 31

Problem 1: A whistler wave propagating at angle $\theta = 60^0$ to static magnetic field $B_S\,\hat{z}$ has $k = (\omega_P / c)(\omega / \omega_c \cos\theta)^{1/2}$. What angle would the group velocity make with $\hat{z}$?

Solution :

The dispersion relation, on writing $\cos\theta = k_z / k$, can be written as

$$k\, k_z\, c^2 = \omega_P^2 \frac{\omega}{\omega_c}$$

Taking $\vec{k} = k_x\,\hat{x} + k_z\,\hat{z},\ k = (k_x^2 + k_z^2)^{1/2}$, the components of group velocity are :

$$v_{gx} = \frac{\partial\omega}{\partial k_x} = \frac{\omega_c}{\omega_P^2}\frac{k_x\, k_z\, c^2}{k},$$

$$v_{gz} = \frac{\partial\omega}{\partial k_z} = \frac{\omega_c}{\omega_P^2}\frac{(k^2 + k_z^2)\, c^2}{k}.$$

The angle group velocity $\vec{v}_g$ makes with $\hat{z}$ is

$$\theta_g = \tan^{-1}\frac{v_{gx}}{v_{gz}} = \tan^{-1}\left(\frac{\sin\theta\cos\theta}{1+\cos^2\theta}\right)$$

$$= \tan^{-1}(0.34) \simeq 20^0.$$

Problem 2: An electron beam propagates along static magnetic field $B_S\,\hat{z}$ in a plasma with velocity $\vec{v}_0 = (c/10)\,\hat{z}$. Obtain the frequency ω (in terms of ω_P and ω_c) of the low frequency whistler, propagating at angle θ to $\hat{z}$, it would excite via Cerenkov resonance.

Solution :

For a low frequency whistler

$$k\,k_z = \frac{\omega_P^2}{c^2}\left(\frac{\omega}{\omega_c}\right), k_z = k\cos\theta.$$

Cerenkov resonance, $\omega = k_z\, v_0$, *gives*

$$k_z^2 = \frac{\omega^2}{v_0^2} = \frac{\omega_P^2}{c^2}\frac{\omega}{\omega_c}\cos\theta$$

$$\omega = \frac{v_0^2}{c^2}\frac{\omega_P^2}{\omega_c}\cos\theta = 10^{-2}\,\frac{\omega_P^2}{\omega_c}\,\cos\theta.$$

Lecture 32

Problem 1: An electrostatic lower hybrid wave of frequency $\omega = 10^{10}$ rad/s, parallel phase velocity $\omega/k_z = c/2$ and the component of power flow density along the magnetic field $B_S\,\hat{z}$ equal to 100 MW/m^2 propagates in a deuterium plasma of electron density 10^{20} m^{-3} and B_s = 4T. Estimate the electric field of the wave.

Solution :

$\omega_P = 5 \times 10^{11}$ rad/s $\omega_c = 6.4 \times 10^{11}$ rad/s. $m_i / m = 3672$`. The effective plasma permittivity for the electrostatic lower hybrid wave is

$$\in = 1 + \frac{\omega_P^2}{\omega_c^2} - \frac{\omega_{Pi}^2}{\omega^2}\left(1 + \frac{k_z^2}{k^2}\frac{m_i}{m}\right) = 0$$

$$\frac{\partial \in}{\partial \omega} = \frac{2}{\omega}\left(1 + \frac{\omega_P^2}{\omega_c^2}\right),$$

$$\nu_{gz} = \frac{\partial \omega}{\partial k_z} \cong \frac{\omega_P^2/\omega^2}{1 + \omega_P^2/\omega_c^2}\frac{k_z\,\omega}{k^2},$$

$$k = k_z\left(\frac{m_i}{m}\right)^{1/2} \Bigg/ \left(\frac{\omega^2(1+\omega_P^2/\omega_c^2)}{\omega_{Pi}^2} - 1\right)^{1/2} = 52\,k_z$$

$$S_{avz} = \frac{\omega}{4}\frac{\partial \in}{\partial \omega} \in_0 |E|^2 \nu_{gz} = 10^8\ W/m^2$$

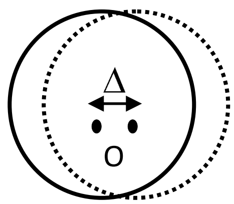
$$|E| = \frac{6.5 \times 10^9\,\omega}{\omega_P}\left(\frac{k^2}{2k_z\omega}\right)^{1/2} \simeq 4 \times 10^5\,V/m.$$

Lecture 41

Problem 1 :A cold plasma rod (nano tube) of radius r_c, having axis along $\hat{z}$, contains singly ionized ions of density n_0 and electrons of same density. The electron cloud is shifted with respect to the ion cloud by a small distance $\Delta\hat{x}$. Obtain the electric field in the overlap region. The electron cloud is now released. Obtain the frequency of oscillation.

Solution :

The axis of the ion cylinder passes through the origin and electron cylinder through (Δ, 0, 0).



The electric field at point $\vec{r} = x\,\hat{x} + y\,\hat{y}$ in the overlap region, due to the ion cylinder of charge density $\rho = n_0 e$ is

$$\vec{E}_1 = \frac{n_0\, e}{2\epsilon_0}\,\Delta\,\hat{x}.$$

The field at $\vec{r}$ due to the electron cylinder is

$$\vec{E}_2 = \frac{-n_0\, e}{2\epsilon_0}\,(\vec{r} - \vec{\Delta})$$

The net electric field in the overlap region is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{n_0\, e}{2\epsilon_0}\,\Delta\,\hat{x}$$

which is uniform. The equation of motion of an electron (of the overlap region) under this field is

$$\frac{d^2\Delta}{dt^2} = -\frac{e\,E_x}{m} = -\frac{n_o\, e^2}{2\epsilon_0\, m}\,\Delta = -\frac{\omega_P^2}{2}\,\Delta.$$

This gives the oscillation frequency $\omega = \omega_P/\sqrt{2}$.